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CRITICAL LOADS OF SELF-SUPPORTING CYLINDRICAL SHELL ROOFS

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Abstract—In this paper we study the determination of critical loads for isolated self-supporting cylindrical shell roofs under dead loads and uniform live loads. It appears necessary to introduce in the circumferential direction a form of buckling more complicated than simple sinusoidal waves, and the problem is solved by the energy method with the use of a simplifying hypothesis of Donnell's theory. In order to apply this theory, we present tables which allow values of the critical loads to be calculated directly in many cases. The results are smaller than the corresponding values obtained in a recent study [1]. Some numerical examples close this paper.

1. INTRODUCTION

THE problem of equilibrium stability of self-supporting cylindrical shell roofs is certainly widely discussed in the recent technical literature. A recent investigation performed by Krall and Caligo [1] excels among the many studies on this subject (see [2–6]) because of its formulation and numerical completeness; in this work the problem is posed and solved approximately, making use of the classical energy methods connected with the well-known principle of Dirichlet.

From these noteworthy results we are allowed to infer that the phenomenon of instability always takes place because of a sudden buckling of the directrix with consequent formation of lobes, the more numerous the larger the radius/span ratio, R/L; this applies to shells which are not too long, that is excluding the possibility of structure yielding due to progressive flattening of the directrix (Brazier's effect—see [7–9]).

Denoting by u, v, and w the displacement components (Fig. 1), which characterize the buckling of the shell as given in [1] we have†

$$u = \frac{\pi R}{L}\frac{w_0}{m^2}\cos m\psi \cos\frac{\pi x}{L}$$

$$v = \frac{w_0}{m}\sin m\psi \sin\frac{\pi x}{L} \tag{1}$$

$$w = w_0 \cos m\psi \sin\frac{\pi x}{L}$$

considering only the case of an even number "m" of half-waves.

In (1), the ratio between the coefficient of tangential components u and v, and the coefficient of normal component w is arbitrary, but chosen in accordance with the logical

† Expressions (1), different from (23) of Note [1], can become similar to the latter if we set:

$$n = 1 \qquad w_0 = A\frac{mL}{\pi R}$$

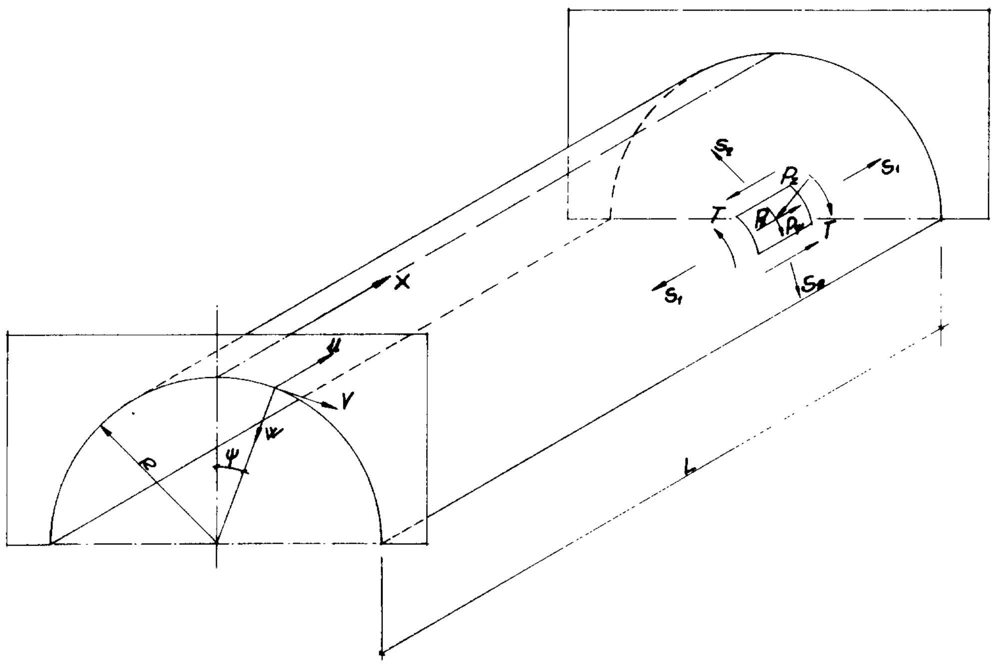

FIG. 1. Geometry of the shell and stress condition.

principle of rendering the extensional energy W_e as small as possible, by annulling the extensional strains $\varepsilon_{\psi}^{(1)}$ and $\gamma_{x\psi}^{(1)}$.

If we accept this principle we must immediately realize, with the help of numerical data of Tables contained in [1], that the normal component w must prevail unavoidably over the tangential components u and v. This appears clear when we consider the expressions (1), bearing in mind that in most cases the buckling takes place with a very high number of half-waves ($m \geq 4$). These remarks are essential in consideration of what follows. We want to consider now for a moment the expressions (1) and we observe that, based on such assumption, buckling develops without change along the whole directrix, thus affecting in the same way the tensile areas and the areas compressed by loads; that is, in vague but efficacious words, the stabilizing and destabilizing areas in the static condition of the structure equilibrium.

It seems reasonable to expect, on the contrary, as clearly confirmed by the results, a buckling prevailingly limited to the most compressed areas of the surface and, consequently, remarkably damped out when proceeding from the top toward the two edge generatrices.† In fact, this appears evident from the study of a cylinder having complete circular directrix compressed and bent by a load $\dot{N}$ capable to generate in the section tensile and compressed areas due to the membrane stress S_1. Figure 2, which appears in Flügge's book [10], clearly shows the above mentioned feature.

Undoubtedly, we must observe that the static condition of self-supporting shell roofs, even if simplified by considering only the membrane stresses, appears to be very different

† Specific reference is made to the case of loads acting in the same direction of the normal inside the top generatrix.

from the elementary condition of the compressed and bent tube shown in Fig. 2. In fact, of essential importance are not only the longitudinal stresses S_1, but also the cross stresses S_2 and the shearing stresses T. We also must face the main problem of expressing, with

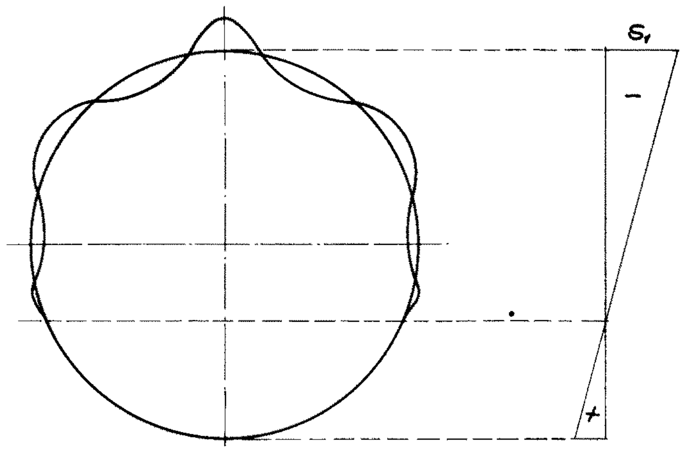

FIG. 2. Buckling shape of a thin cylinder eccentrically compressed.

less restrictions than (1), the buckling of the shell when passing from the ideal non-buckled to the buckled state typical of the neutral equilibrium. Obviously, such a problem can be solved by the traditional method as indicated by [1] in the case of a simple sinusoidal waves buckling, but can be further simplified if we accept the correct deductions drawn by the results of [1]. Among the other deductions, a very important one is that concerning the great number of half-waves typical of the buckling of the surface when in neutral equilibrium, since the other deduction (prevalence of w over u and v) ensues directly from the first one.

Under these conditions we can accept the remarkable simplifications suggested by Donnell, which lead to the well-known equations (see [11–14]) bearing Donnell's name. We do not discuss the acceptability of such simplifications, since this is a well-known subject [15]; on the contrary, we will prove its validity also for the specific case we are studying, making use of extensive numerical analysis. However, we would like to point out the noteworthy advantages of such simplifications in comparison with the traditional method: the considerable reduction of numerical operations, and, above all, the possibility of carrying out an approximate analysis of critical loads by choosing arbitrarily only the normal component w, in lieu of all the components u, v, w.

2. APPLICATION OF THE ENERGY METHOD TO THE ANALYSIS OF CRITICAL LOADS OF SELF-SUPPORTING SHELL ROOFS IN ACCORDANCE WITH DONNELL'S THEORY

C_0 is the equilibrium configuration (ideally non-deformed) of the shell subject to a load represented by components p_x, p_ψ, p_z. Assuming the hypothesis of membrane behaviour, the internal stresses S_1, S_2, T can be obtained by solving the well-known equations:

$$\begin{aligned} \frac{\partial S_1}{\partial x}+\frac{1}{R}\frac{\partial T}{\partial \psi}+p_x &= 0 \\ \frac{\partial T}{\partial x}+\frac{1}{R}\frac{\partial S_2}{\partial \psi}+p_\psi &= 0 \\ \frac{S_2}{R}+p_z &= 0. \end{aligned} \tag{2}$$

If $p_z = 0$, on the two extreme generatrices, to such stresses we must add, for the equilibrium, two forces Z_1 and Z_2 absorbed by filiform elements placed near the two above-mentioned generatrices, in order to eliminate the possible shearing stresses T resulting from the solution of equations (2).

Such forces are determined by the following two relations:

$$\begin{aligned} \frac{\mathrm{d}Z_1}{\mathrm{d}x}-T\left(x,\frac{\pi}{2}\right) &= 0 \\ \frac{\mathrm{d}Z_2}{\mathrm{d}x}+T\left(x,-\frac{\pi}{2}\right) &= 0 \end{aligned} \tag{3}$$

and, if considered with the stresses deriving from (2), they express the static condition by which the loads affecting the surface are transmitted by the latter to the supports. This type of equilibrium, which is always possible when the deformation of the surface is negligible, can loose its stability when external loads reach a certain value and becomes neutral assuming a new shape very close to the ideal undeformed state.

Since C_1 is a general deformed state, it expresses only neutral equilibrium and only in the case of a surface in perfect equilibrium as it was in the undeformed state C_0. This gives the possibility of establishing certain relations among the displacement components u, v, w, which express the change of the structure from C_0 to C_1.

Under particular conditions, with the assumption of some non-restrictive hypotheses on the secondary condition C_1, it is possible, with some acceptable simplifications, to connect two of the above-mentioned relations; this way we can express directly two of the three above-mentioned components as linear functions of the third one. This happens just in the case we are studying, under the hypothesis of a large number of half-waves in the buckled state C_1, in the case of neutral equilibrium.

In fact, u, v, w being the displacement components of C_1, for the first order strain components we will have:

$$\begin{aligned} \varepsilon_x^{(1)} &= \frac{\partial u}{\partial x} = \frac{1}{Eh}(\delta S_1 - \nu\delta S_2) \\ \varepsilon_\psi^{(1)} &= \frac{1}{R}\left(\frac{\partial v}{\partial \psi}-w\right) = \frac{1}{Eh}(\delta S_2 - \nu\delta S_1) \\ \gamma_{x\psi}^{(1)} &= \frac{1}{R}\frac{\partial u}{\partial \psi}+\frac{\partial v}{\partial x} = \frac{2(1+\nu)}{Eh}\delta T \end{aligned} \tag{4}$$

where E and ν are, respectively, the modulus of elasticity and Poisson's ratio of the material forming the shell; and δS_1, δS_2, δT are the membrane stresses variations caused by the change $C_0 \rightarrow C_1$. For these latters we can assume, in accordance with Donnell, that the equilibrium conditions along generatrices (x) and directrices (ψ) can still be expressed by:

$$\begin{aligned} \frac{\partial(\delta S_1)}{\partial x}+\frac{1}{R}\frac{\partial(\delta T)}{\partial \psi} &= 0 \\ \frac{\partial(\delta T)}{\partial x}+\frac{1}{R}\frac{\partial(\delta S_2)}{\partial \psi} &= 0 \end{aligned} \tag{5}$$

which are formally similar to the first two equations (2) and expressed as if the element was not buckled and flexural stresses were not present. This assumption, with (4) and (5), will yield the following two linear differential equations:

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2}+\frac{1-\nu}{2R^2}\frac{\partial^2 u}{\partial \psi^2}+\frac{1+\nu}{2R}\frac{\partial^2 v}{\partial x\,\partial \psi} &= \frac{\nu}{R}\frac{\partial w}{\partial x} \\ \frac{1}{R^2}\frac{\partial^2 v}{\partial \psi^2}+\frac{1-\nu}{2}\frac{\partial^2 v}{\partial x^2}+\frac{1+\nu}{2R}\frac{\partial^2 u}{\partial x\,\partial \psi} &= \frac{1}{R^2}\frac{\partial w}{\partial \psi} \end{aligned} \tag{6}$$

which lead to the expressions of components u and v, after determining the expression of normal component w. We can further simplify the problem, only formally now, by assuming as unknown, in lieu of displacement components u and v, the stress function $F(x, \psi)$ connected with the internal stresses by:

$$\begin{aligned} \delta S_1 &= \frac{1}{R^2}\frac{\partial^2 F}{\partial \psi^2} \\ \delta T &= -\frac{1}{R}\frac{\partial^2 F}{\partial x\,\partial \psi} \\ \delta S_2 &= \frac{\partial^2 F}{\partial x^2} \end{aligned} \tag{7}$$

which satisfy identically the equilibrium conditions (5), and we obtain the relation between $F(x, \psi)$ and $w(x, \psi)$ from the compatibility condition (see [5]):

$$\frac{1}{R^2}\frac{\partial^2 \varepsilon_x^{(1)}}{\partial \psi^2}+\frac{\partial^2 \varepsilon_\psi^{(1)}}{\partial x^2}-\frac{1}{R}\frac{\partial^2 \gamma_{x\psi}^{(1)}}{\partial x\,\partial \psi} = -\frac{1}{R}\frac{\partial^2 w}{\partial x^2} \tag{8}$$

which, in fact, expressed by the function $F(x, \psi)$ with (4) and (7) gives:

$$\frac{\partial^4 F}{\partial x^4}+\frac{2}{R^2}\frac{\partial^4 F}{\partial x^2\,\partial \psi^2}+\frac{1}{R^4}\frac{\partial^4 F}{\partial \psi^4} = -\frac{Eh}{R}\frac{\partial^2 w}{\partial x^2}. \tag{9}$$

In this manner we have satisfied, at least approximately, the equilibrium conditions for the displacement with respect to generatrices and directrices of the shell element in buckled condition C_1; now we can deduce further equilibrium conditions not directly, that is geometrically, but with a variation procedure by imposing the extremum condition for the function:

$$\Phi = W+L_2^*-L_2 \tag{10}$$

which physically coincides with the second variation of total potential energy when passing from state C_0 to state C_1.

The terms forming Φ are, respectively, the elastic energy W, the second order work L_2^* of internal stresses S_1, S_2, T, and internal forces Z_1, Z_2, and the second order work L_2 of external loads p_x, p_ψ, p_z.

For this latter we must observe that, for the loading under examination, we will always have:

$$L_2 = 0. \tag{11}$$

Furthermore, it must be noticed that for the second order work L_2^*, actually the state of stress of the shell in the main unbuckled state C_0 can be substantially different from the state which is obtained through the simple hypothesis of membrane behaviour.

Even the extensional stresses S_1, S_2, T have a much more irregular distribution than in the hypothesis of membrane behaviour, and, moreover, the bending moments M_1 and M_2, as well as the twisting moment H are present. For these latter, which would do work because of the second order curvature changes, it was demonstrated that the internal work L_{2f}^* related to the work of extensional stresses L_{2e}^* depends upon the ratio h/R and therefore, since h/R does not exceed 2/100, is negligible [16].

Still to be resolved is the main problem concerning the internal work L_{2e}^*, which should be calculated by taking into account the actual extensional stresses S_1, S_2, T rather than those obtained through the simple hypothesis of membrane behaviour. However, it must be observed that, if we add to the work of the membranal stresses S_1, S_2, T the work of balancing forces Z_1, Z_2, which is obviously not present if the actual values of S_1, S_2, T are used, the value of extensional work L_{2e}^* so calculated is very close to the value of the work calculated through the actual stresses. This can be easily justified, even by intuition, if we observe that the high values of the actual internal stress S_1 at the edges justify Z_1 and Z_2 as impulsive and therefore can be considered as concentrated forces at the edges.

Since in this study we will consider a shell having semi-circular directrix, subject to dead load g and to uniform overload on the horizontal plane p, the above considerations are obviously necessary only for the tensile state due to dead load g, for which the internal forces Z_1, Z_2 are different from zero. In fact, in the case of live overload p the vanishing of Z_1, Z_2 confirms with sufficient accuracy the validity of the hypothesis of membrane behaviour of the shell, and therefore the above considerations are of no use. Thus the results which will be obtained are to be considered exact in the case of live overload and sufficiently approximate in the case of dead load g.

2.1 *Elastic energy expressions*

The elastic energy W in the sum:

$$W = W_f + W_e \tag{12}$$

is expressed by W_f, flexural elastic energy, and W_e, extensional elastic energy.† We have

† In reality, it is demonstrated (see [6]) that, because of the directrices curvature we have a mixed term W_{ef} which is present in the elastic energy W of shell. In most cases such a term is quite insignificant compared to the extensional elastic energy W_e and the flexural energy W_f, and we can neglect it.

$$W_f = \frac{Eh^3}{24(1-\nu^2)} \int_\Omega \{(\chi_x^{(1)} + \chi_\psi^{(1)})^2 + 2(1-\nu)(\overset{2}{\bar{\chi}}{}_{x\psi}^{(1)} - \chi_x^{(1)}\chi_\psi^{(1)})\} \, d\Omega$$

$$W_e = \frac{Eh}{2(1-\nu^2)} \int_\Omega \{(\varepsilon_x^{(1)} + \varepsilon_\psi^{(1)})^2 + \frac{1-\nu}{2}(\overset{2}{\bar{\gamma}}{}_{x\psi}^{(1)} - 4\varepsilon_x^{(1)}\varepsilon_\psi^{(1)})\} \, d\Omega \tag{13}$$

where

$$d\Omega = R \, dx \, d\psi \tag{14}$$

is a surface element of the shell; $\chi_x^{(1)}$, $\chi_\psi^{(1)}$, $\chi_{x\psi}^{(1)}$ the flexural and torsional curvature changes of the first order of the element; and $\varepsilon_x^{(1)}$, $\varepsilon_\psi^{(1)}$, $\gamma_{x\psi}^{(1)}$ the components of the first order of extensional strains already considered in (4).

In accordance with Donnell for the first ones we have:

$$\chi_x^{(1)} = \frac{\partial^2 w}{\partial x^2}$$

$$\chi_\psi^{(1)} = \frac{1}{R^2}\frac{\partial^2 w}{\partial \psi^2} \tag{15}$$

$$\chi_{x\psi}^{(1)} = \frac{1}{R}\frac{\partial^2 w}{\partial x \, \partial \psi}.$$

Therefore, the flexural elastic energy is expressed as follows:

$$W_f = \frac{Eh^3}{24(1-\nu^2)} \int_\Omega \left\{ \left(\frac{\partial^2 w}{\partial x^2} + \frac{1}{R^2}\frac{\partial^2 w}{\partial \psi^2} \right)^2 + \frac{2(1-\nu)}{R^2} \left[\left(\frac{\partial^2 w}{\partial x \, \partial \psi} \right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial \psi^2} \right] \right\} d\Omega \tag{16}$$

and in w it appears as a quadratic form.

For the extensional elastic energy, in view of the use of function F, it is advisable to obtain with equations (4) the following variation to the second equation (13):

$$W_e = \frac{1}{2Eh} \int_\Omega \left\{ (\delta S_1 + \delta S_2)^2 + \frac{2(1+\nu)}{R^2}[(\delta T)^2 - \delta S_1 \, \delta S_2] \right\} d\Omega \tag{17}$$

Because of (7), we obtain the final expression of W_e from equation (17):†

† In expressing the extensional energy we should consider W_e^* absorbed by edge tension bars, which would be expressed as follows when such tension bars are made or iron rods having an area A_{f1} and A_{f2} respectively:

$$W_e^* = \tfrac{1}{2}E_f A_{f1} \int_0^L \varepsilon_{f1}^{(1)} \, dx + \tfrac{1}{2}E_f A_{f2} \int_0^L \varepsilon_{f2}^{(1)} \, dx \tag{18}$$

where E_f is the elasticity modulus of the steel used, and $\varepsilon_{f1}^{(1)}$ and $\varepsilon_{f2}^{(1)}$ the corresponding strains of the first order, which can be calculated with the following equations:

$$\varepsilon_{f1}^{(1)} = \frac{1}{Eh}\left(\frac{1}{R^2}\frac{\partial^2 F}{\partial \psi^2} - \nu \frac{\partial^2 F}{\partial x^2} \right)_{\psi = \pi/2}$$

$$\varepsilon_{f2}^{(1)} = \frac{1}{Eh}\left(\frac{1}{R^2}\frac{\partial^2 F}{\partial \psi^2} - \nu \frac{\partial^2 F}{\partial x^2} \right)_{\psi = -\pi/2}. \tag{19}$$

Actually, the energy (18) appears to be quite insignificant compared to the energy (17); thus we can neglect it.

$$W_e = \frac{1}{2E}\int_\Omega \left\{\left(\frac{\partial^2 F}{\partial x^2}+\frac{\partial^2 F}{R^2\partial\psi^2}\right)^2 + \frac{2(1+\nu)}{R^2}\left[\left(\frac{\partial^2 F}{\partial x\,\partial\psi}\right)^2 - \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 F}{\partial\psi^2}\right]\right\} d\Omega \tag{20}$$

and we must note that, since we can express F as a function of w with equation (9), even equation (20) appears as a quadratic form in w.

2.2 *Second order work expressions*

The second order work, limited in our case to L_2^*, is generally expressed as follows:

$$L_2^* = \int_\Omega \{S_1\varepsilon_x^{(2)} + T\gamma_{x\psi}^{(2)} + S_2\varepsilon_\psi^{(2)}\}\, d\Omega + \int_0^L Z_1\varepsilon_x^{(2)}\left(x, \frac{\pi}{2}\right) dx + \int_0^L Z_2\varepsilon_x^{(2)}\left(x, -\frac{\pi}{2}\right) dx \tag{21}$$

where $\varepsilon_x^{(2)}$, $\varepsilon_\psi^{(2)}$, $\gamma_{x\psi}^{(2)}$ are the second order components of the extensional strains of the shell.

In accordance with Donnell, we can assume:

$$\begin{aligned} \varepsilon_x^{(2)} &= \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \\ \varepsilon_\psi^{(2)} &= \frac{1}{2R^2}\left(\frac{\partial w}{\partial \psi}\right)^2 \\ \gamma_{x\psi}^{(2)} &= \frac{1}{R}\frac{\partial w}{\partial x}\frac{\partial w}{\partial \psi} \end{aligned} \tag{22}$$

and we obtain the following final expression for the second order work:

$$L_2^* = \frac{1}{2}\int_\Omega \left\{S_1\left(\frac{\partial w}{\partial x}\right)^2 + \frac{2T}{R}\frac{\partial w}{\partial x}\frac{\partial w}{\partial \psi} + \frac{S_2}{R^2}\left(\frac{\partial w}{\partial \psi}\right)^2\right\} d\Omega + \frac{1}{2}\int_0^L Z_1\left(\frac{\partial w}{\partial x}\right)^2_{\psi=\pi/2} dx + \frac{1}{2}\int_0^L Z_2\left(\frac{\partial w}{\partial x}\right)^2_{\psi=-\pi/2} dx. \tag{23†}$$

L_2^* is also expressed by a quadratic form of w. Now we can calculate the critical loads of the shell for various load conditions.

† Also in this case equation (23) is approximate, inasmuch as the second order work is actually the sum

$$L_2^* = L_{2e}^* + L_{2f}^* + L_{2ef}^* + L_{2fe}^* \tag{21'}$$

where the first two terms represent the work of extensional and flexural stresses for second order extensional and flexural strains, and the other two, due to the directrix curvature, represent the mixed terms caused by the fact that stresses are not orthogonal to the strain components (see [6]). On the other hand, if we limit, as in our case, the definition of the static equilibrium to the membrane stresses, having

$$L_{2f}^* = L_{2fe}^* = 0$$

equation (21)' can be reduced to

$$L_2^* = L_{2e}^* + L_{2ef}^*$$

Thus the approximation is limited only to the term L_{2ef}^* which appears negligible compared to the first one.

Actually, in [6] it is clear that the correct calculation of L_2^* through the actual stresses of flexural solution of the shell and the complete expression (21)' affects but very slightly the critical loads; therefore the use of equation (21) appears sufficiently approximate.

The variation of equation (10) with respect to w, in accordance with (16), (20), and (23), gives the Eulerian differential equation:

$$\frac{\partial^4 w}{\partial x^4}+\frac{2}{R^2}\frac{\partial^4 w}{\partial x^2\,\partial\psi^2}+\frac{1}{R^4}\frac{\partial^4 w}{\partial\psi^4}-\frac{1}{DR}\frac{\partial^2 F}{\partial x^2}=\frac{1}{D}\left(S_1\frac{\partial^2 w}{\partial x^2}+\frac{2T}{R}\frac{\partial^2 w}{\partial x\,\partial\psi}+\frac{S_2}{R^2}\frac{\partial^2 w}{\partial\psi^2}\right)$$

$$\left[D=\frac{Eh^3}{12(1-\nu^2)}\right] \tag{24}$$

with the boundary conditions for a general case.

Equation (24), together with equation (9), is the basic system governing, in accordance with Donnell, the problem of elastic equilibrium of thin cylindrical shells. It is also possible to obtain a single differential equation by expressing $\partial^2 F/\partial x^2$ as a function of w with equation (24) and substituting it into equation (9) and differentiating with respect to x. This leads to the equation:

$$D\Delta\Delta\Delta\Delta w-\frac{h}{R^2}\frac{\partial^4 w}{\partial x^4}=\Delta\Delta\left(S_1\frac{\partial^2 w}{\partial x^2}+\frac{2T}{R}\frac{\partial^2 w}{\partial x\,\partial\psi}+\frac{S_2}{R^2}\frac{\partial^2 w}{\partial\psi^2}\right) \tag{25}$$

where:

$$\Delta=\frac{\partial^2}{\partial x^2}+\frac{1}{R^2}\frac{\partial^2}{\partial\psi^2}, \tag{26}$$

which is known as Donnell's equation.

In our case we cannot solve in closed form equation (25) or the system (9), (24); therefore we will use the approximate solution of the buckling problem, giving an appropriate expression to the normal component w and obtaining the critical loads from the solution of the extremum problem:

$$\delta\Phi = 0 \tag{27}$$

for each small variation of the parameters left unknown in w.

3. CALCULATION OF CRITICAL LOADS FOR SEMI-CIRCULAR ISOLATED SHELL ROOFS

As indicated in the introduction, one of the main objects of this research work is to investigate how the choice of a buckling shape affects the critical loads. The choice is now limited to the component w, which will be taken as

$$w = w_0(1+\eta\cos 2\psi)\cos m\psi \sin \pi x/L. \tag{28}$$

Equation (28) is different from the last of equations (1) because of the correction factor $(1+\eta\cos 2\psi)$ which appears to be of basic importance for the numerical determination of critical loads in the case of dead load as well as, and even more, in the case of uniform live load at the base. In fact, it is sufficient to notice that equation (28), even including the case of simple sinusoidal waves buckling for $\eta = 0$, permits the amplitude of the directrix waves to decrease considerably by moving from the top toward the edge generatrices in accordance with numerical values of parameter η.

We will limit the analysis, as in [1], to the case where m is even, excluding the case $m = 2$, from the examination of tables contained in [1], appears to be quite insignificant.†

Function $F(x, \psi)$ with equation (9) can be determined through the following equation:

$$\frac{\partial^4 F}{\partial x^4}+\frac{2}{R^2}\frac{\partial^4 F}{\partial x^2\,\partial\psi^2}+\frac{1}{R^4}\frac{\partial^4 F}{\partial\psi^4} = \frac{\pi^2 Eh}{L^2 R}w_0(1+\eta\cos 2\psi)\cos m\psi\sin\frac{\pi x}{L} \tag{29}$$

and corresponding boundary conditions on extreme directrices and generatrices.

In the case of an isolated shell, the boundary conditions on the two extreme directrices are expressed as, with the known hypothesis of unextensional transverses unable to absorb stresses normal to their plane:

$$v = 0, \qquad \delta S_1 = 0 \qquad (x = 0, x = L) \tag{30}$$

and are expressed as follows:

$$F = \frac{\partial^2 F}{\partial x^2} = 0 \qquad (x = 0, x = L). \tag{31}$$

The boundary conditions on two extreme generatrices are expressed as follows, neglecting the effects of edge reinforcement bars:

$$\delta S_2 = 0, \qquad \delta T = 0 \qquad \left(\psi = \pm\frac{\pi}{2}\right) \tag{32}$$

and are therefore expressed by the following relations:

$$\frac{\partial^2 F}{\partial x^2} = \frac{\partial^2 F}{\partial x\,\partial\psi} = 0 \qquad \left(\psi = \pm\frac{\pi}{2}\right). \tag{33}‡$$

† In fact, from the tables of [1] it appears that only in one case does the shell buckle with a double half-wave and, more specifically, in the case of live load condition p and for the geometrical ratios $R/L = 0{\cdot}01$ $h/R = 0{\cdot}02$. In this study the case $m = 2$ has been examined separately.

‡ Actually, without neglecting the effect of the reinforcement bars, the second relation (33) should be written as:

$$\frac{d(\partial Z_1)}{dx}-\partial T\left(x,\frac{\pi}{2}\right) = 0, \qquad \frac{d(\partial Z_2)}{dx}+\partial T\left(x,-\frac{\pi}{2}\right) = 0 \tag{34}$$

where

$$\partial Z_1 = E_f A_{f1}\varepsilon_{f1}^{(1)}, \qquad \partial Z_2 = E_f A_{f2}\varepsilon_{f1}^{(1)}$$

the variations of stresses developing during the passage $C_0 \to C_1$. In consideration of (7) and (22) the second of equations (33) should be expressed as follows:

$$\frac{\partial}{\partial x}\left\{\frac{E_f A_{f1}}{Eh}\left(\frac{1}{R^2}\frac{\partial^2 F}{\partial\psi^2}-\nu\frac{\partial^2 F}{\partial x^2}\right)-\frac{1}{R}\frac{\partial F}{\partial\psi}\right\} = 0 \qquad \left(\psi = \frac{\pi}{2}\right) \tag{35}$$

$$\frac{\partial}{\partial x}\left\{\frac{E_f A_{f2}}{Eh}\left(\frac{1}{R^2}\frac{\partial^2 F}{\partial\psi^2}-\nu\frac{\partial^2 F}{\partial x^2}\right)+\frac{1}{R}\frac{\partial F}{\partial\psi}\right\} = 0 \qquad \left(\psi = -\frac{\pi}{2}\right) \tag{36}$$

Actually, the exiguity of ratio:

$$\rho = E_f A_f/EhR \tag{37}$$

permits to study the problem with sufficient approximation using the second of equations (33) in lieu of the more correct (35) and (36) which can be reduced to the first one for vanishing ρ.

With boundary conditions (31) and (33), equation (29) gives for F:

$$F(x,\psi) = \frac{EL^2\alpha}{\pi^2}w_0\left\{\frac{1}{K_0^2}(\cos m\psi + f_0\cosh\beta\psi + f_1\beta\psi\sinh\beta\psi)\right.$$

$$+\frac{\eta}{2K_1^2}[\cos(m-2)\psi - f_0\cosh\beta\psi - f_1\beta\psi\sinh\beta\psi] \tag{38}$$

$$\left.+\frac{\eta}{2K_2^2}[\cos(m+2)\psi - f_0\cosh\beta\psi - f_1\beta\psi\sinh\beta\psi]\right\}$$

where

$$\alpha = \frac{h}{R}, \qquad \beta = \pi\frac{R}{L}$$

$$K_0 = 1+\frac{m^2}{\beta^2}, \qquad K_1 = 1+\frac{(m-2)^2}{\beta^2}, \qquad K_2 = 1+\frac{(m+2)^2}{\beta^2} \tag{39}$$

$$f_0 = -2(-1)^{m/2}\frac{\sinh\frac{\pi\beta}{2}+\frac{\pi\beta}{2}\cosh\frac{\pi\beta}{2}}{\pi\beta+\sinh\pi\beta}, \qquad f_1 = 2(-1)^{m/2}\frac{\sinh\frac{\pi\beta}{2}}{\pi\beta+\sinh\pi\beta}$$

Now we know the stress function F and we can calculate the extensional and flexural elastic energy of the shell. In fact, with

$$\varphi = \frac{8}{\pi\beta}\frac{\cosh\pi\beta - 1}{\sinh\pi\beta+\pi\beta} \tag{40}$$

from equations (38) and (20) we obtain for the extensional elastic energy:

$$W_e = \frac{\pi EL}{8}\alpha w_0^2\left\{\frac{1}{K_0^2}\left(1-\frac{\varphi}{K_0^2}\right)+\eta\frac{\varphi}{K_0^2}\left(\frac{1}{K_1^2}+\frac{1}{K_2^2}\right)\right.$$
$$\left.+\frac{\eta^2}{4}\left(\frac{1}{K_1^2}+\frac{1}{K_2^2}\right)\left[1-\varphi\left(\frac{1}{K_1^2}+\frac{1}{K_2^2}\right)\right]\right\} \tag{41}$$

while, from equations (28) and (16) we obtain for flexural elastic energy:

$$W_f = \frac{\pi EL}{96(1-\nu^2)}\alpha^3\beta^4 w_0^2\left\{K_0^2+\frac{\eta^2}{4}(K_1^2+K_2^2)\right\}. \tag{42}$$

Then, with

$$\sigma_0 = \left\{3(1-\nu^2)\frac{\alpha}{K_0^2}\left(1-\frac{\varphi}{K_0^2}\right)+\frac{\alpha^3\beta^4}{4}K_0^2\right\}.10^6$$

$$\sigma_1 = 3(1-\nu^2)\varphi\frac{\alpha}{K_0^2}\left(\frac{1}{K_1^2}+\frac{1}{K_2^2}\right).10^6 \tag{43}$$

$$\sigma_2 = \left\{3(1-\nu^2)\frac{\alpha}{4}\left(\frac{1}{K_1^2}+\frac{1}{K_2^2}\right)\left[1-\varphi\left(\frac{1}{K_1^2}+\frac{1}{K_2^2}\right)\right]+\frac{\alpha^3\beta^4}{16}(K_1^2+K_2^2)\right\}.10^6$$

the total elastic energy of the shell can be expressed as:

$$W = \frac{\pi}{24} \cdot \frac{E \,.\, 10^{-6}}{1-\nu^2}(\sigma_0+\sigma_1\eta+\sigma_2\eta^2)L \,.\, w_0^2. \tag{44}$$

We must now determine the second order work L_2^* for the two loading conditions examined.

In the case of dead load g the membrane stresses relative to initial state C_0 are represented by the well known expressions:

$$\begin{aligned} S_1 &= -\frac{gL^2}{R}\left(\frac{x}{L}-\frac{x^2}{L^2}\right)\cos\psi \\ T &= gL\left(1-\frac{2x}{L}\right)\sin\psi \\ S_2 &= -gR\cos\psi \end{aligned} \tag{45}$$

to which we add, for the necessary equilibrium conditions, the tensile forces:

$$Z_1 = Z_2 = gL^2\left(\frac{x}{L}-\frac{x^2}{L^2}\right). \tag{46}$$

Therefore, if we set as in [1]:

$$g = \frac{\pi}{24} \cdot \frac{E \,.\, 10^{-6}}{1-\nu^2}\lambda_g \tag{47}$$

we obtain from equations (23) and (28):

$$L_{2g}^* = -\frac{\pi}{24} \cdot \frac{E \,.\, 10^{-6}}{1-\nu^2}(\gamma_{0g}+\gamma_{1g}\eta+\gamma_{2g}\eta^2)Lw_0^2\lambda_g \tag{48}$$

where

$$\begin{aligned} \gamma_{0g} &= \frac{m^2}{4m^2-1}\left(m^2-1-\frac{\pi^2+3}{6}\right) \\ \gamma_{1g} &= \frac{1}{6}\left(m^2+4+\frac{7\pi^2+3}{6}\right)+\frac{1}{4(m-1)^2-1}\left[(m-1)^2-\frac{m}{4}(m-2)\right. \\ &\quad \left.+\frac{\pi^2+3}{24}\right]+\frac{1}{4(m+1)^2-1}\left[(m+1)^2-\frac{m}{4}(m+2)+\frac{\pi^2+3}{24}\right] \\ \gamma_{2g} &= \frac{1}{60}\left(7m^2+24-23\frac{\pi^2+3}{6}\right)-\frac{1}{4m^2-1}\left(\frac{3m^2+4}{8}+\frac{\pi^2+3}{48}\right) \\ &\quad -\frac{1}{4(m-2)^2-1}\left[3\left(\frac{m-2}{4}\right)^2+\frac{\pi^2+3}{96}\right]-\frac{1}{4(m+2)^2-1}\left[3\left(\frac{m+2}{4}\right)^2+\frac{\pi^2+3}{96}\right]. \end{aligned} \tag{49}$$

In the case of live load p considered to be uniformly distributed in the horizontal plane, the membrane stresses in state C_0 are represented by:

$$S_1 = -\frac{3pL^2}{2R}\left(\frac{x}{L}-\frac{x^2}{L^2}\right)\cos 2\psi$$

$$T = \frac{3pL}{4}\left(1-\frac{2x}{L}\right)\sin 2\psi \tag{50}$$

$$S_2 = -pR\cos^2\psi$$

and here

$$Z_1 = Z_2 = 0. \tag{51}$$

Therefore, if in accordance with [1], we now have:

$$p = \frac{1}{6}\cdot\frac{E\,.\,10^{-6}}{1-\nu^2}\lambda_p. \tag{52}$$

The second order work L_2^* with equations (23) and (28) can be written as follows:

$$L_{2p}^* = -\frac{\pi}{24}\cdot\frac{E\,.\,10^{-6}}{1-\nu^2}(\gamma_{0p}+\gamma_{1p}\eta+\gamma_{2p}\eta^2)Lw_0^2\,.\,\lambda_p \tag{53}$$

where now:

$$\gamma_{0p} = \frac{m^2}{4}$$

$$\gamma_{1p} = \frac{1}{4}\left(m^2+3+\frac{\pi^2+3}{6}\right) \tag{54}$$

$$\gamma_{2p} = \frac{1}{4}(m^2+4).$$

In both cases the second order variation of total potential energy Φ expressed by equation (10) is as follows:

$$\Phi = \frac{\pi}{24}\cdot\frac{E\,.\,10^{-6}}{1-\nu^2}\{\sigma_0+\sigma_1\eta+\sigma_2\eta^2-\lambda(\gamma_0+\gamma_1\eta+\gamma_2\eta^2)\}Lw_0^2 \tag{55}$$

thus the extremum condition:

$$\frac{\partial\Phi}{\partial w_0} = 0 \tag{56}$$

gives:

$$\lambda = \frac{\sigma_0+\sigma_1\eta+\sigma_2\eta^2}{\gamma_0+\gamma_1\eta+\gamma_2\eta^2} \tag{57}$$

while the minimum condition:

$$\frac{d\lambda}{d\eta} = 0 \tag{58}$$

permits the determination by one of the two roots:

$$\eta_{1,2} = \frac{\sigma_2\gamma_0 - \sigma_0\gamma_2}{\sigma_2\gamma_1 - \sigma_1\gamma_2}\left\{-1 \pm \sqrt{\left[1 + \frac{(\sigma_2\gamma_1 - \sigma_1\gamma_2)(\sigma_0\gamma_1 - \sigma_1\gamma_0)}{(\sigma_2\gamma_0 - \sigma_0\gamma_2)^2}\right]}\right\} \tag{59}$$

the value η^* for which the ratio:

$$\lambda^* = \frac{\sigma_0 + \sigma_1\eta^* + \sigma_2\eta^{*2}}{\gamma_0 + \gamma_1\eta^* + \gamma_2\eta^{*2}} \tag{60}$$

represents the minimum value of ratio (57) for every value of m previously determined. The minimum value of λ^* with respect to m, (even integer and higher than two) finally gives the desired critical multiplier λ_{crit}.

We can observe that the expressions (43), (49), and (54) are valid when m is even and different from two. For $m = 2$ such coefficients cannot be obtained by said equations, but must be calculated by repeating the calculation of the energy and second order work. We do not repeat here such results which are completely useless (see [1]) for the numerical work.

4. NUMERICAL ANALYSIS FOR THE CASE OF SIMPLE SINUSOIDAL WAVES BUCKLING

Before starting the calculation of critical multipliers λ_{crit} for the load conditions under consideration, we want to show the acceptability of approximations connected with Donnell's theory for the case we are examining, by means of a numerical analysis of minimum multipliers $\lambda_{0\,crit}$.

Such multipliers, obtained by (57) for $\eta = 0$, represent the minimum with respect to m, of ratio:

$$\lambda_0 = \frac{\sigma_0}{\gamma_0} \tag{61}$$

and necessarily, cannot be too different from λ_{crit} obtained in [1], since in this case the prevailing part of the buckling shape is identical.

The results obtained, shown in Table 1 for the dead load g, and in Table 2 for live load p, clearly confirm our hypotheses. In fact, from Tables 1 and 2, numerically developed with the assumption that $\nu = 0$ in (43), we can obtain as extreme values of the percentage differences compared to the corresponding values shown in Tables II and III of paper [1]:

$$S_{min} = -8{\cdot}5\% \qquad S_{max} = +7\% \tag{62}$$

in case of dead load, and

$$S_{min} = -10\% \qquad S_{max} = 0\% \tag{63}$$

in case of live load.

The above differences are not to be considered errors due to the approximations relative to Donnell's theory, since also comparison values of Tables contained in [1] are approximate. On the contrary, the smallness of such differences confirms that, having the same normal component w, the traditional method with an appropriate, even if arbitrary, choice of components u and v, gives results which are substantially similar to the results obtained with the simpler theory of Donnell. In view of this, the numerical data of Tables 1 and 2 can be used for comparison when we investigate the influence on the critical multiplier of a buckling shape closer to the reality than the simple sinusoidal waves buckling typical of subject results.

5. NUMERICAL ANALYSIS FOR THE CASE OF DAMPED SINUSOIDAL WAVES BUCKLING

Tables 4 and 5 give the critical multipliers λ_{crit} for the same geometrical ratios of Tables 1 and 2 with the procedure outlined at the end of Section 3 above.

Such tables have been numerically developed with the assumption, for the coefficient calculation (43), that $\nu = 0$ and varying m from 4 to 24, two units at a time. The case $m = 2$ has been examined separately.

From a study of these tables the importance of factor η on the numerical value of critical multiplier becomes clear. If we compare them with Tables 1 and 2 we notice a decrease of 30 per cent and more for the multipliers relative to dead load g, and a decrease of 50 per cent and more for the multipliers relative to live load p.

On the contrary, the number m of half-waves expressing the buckled condition C_1 appears substantially unchanged if compared to the number of half-waves obtained in the case of simple sinusoidal waves buckling. The variation of the percentage differences for the multipliers of dead load g and the multipliers of live load p is evident: in fact, since because of g the tensile stresses affect almost exclusively the two edge reinforcement bars, while in the case of p they affect a larger zone, a decrease of the half-waves amplitude is more important in this case than in the previous one. Notwithstanding the remarkable decrease, sometimes the critical loads

$$
\begin{aligned}
g_{\text{crit}} &= \frac{\pi}{24}\,\frac{E\,.\,10^{-6}}{1-\nu^2}\lambda_{g\,\text{crit}} \\
p_{\text{crit}} &= \frac{1}{6}\,\frac{E\,.\,10^{-6}}{1-\nu^2}\lambda_{p\,\text{crit}}
\end{aligned}
\tag{64}
$$

remain so high that we can foresee for them stresses σ_{crit} above the proportionality limit σ_p. Assuming as comparison tension $\sigma_{\text{crit}\,z}$ the maximum value of two principal tensions σ_x and σ_ψ, present in the center cross section in correspondence with the top generatrix of the shell, and

$$
\begin{aligned}
E &= 2\times 10^6\ \text{ton/m}^2 \\
\nu &= 0{\cdot}2 \\
\sigma_p &= 1500\ \text{ton/m}^2
\end{aligned}
\tag{65}
$$

we marked in Tables 3 and 4 the lines limiting the range of critical multipliers for which we have:

$$\sigma_{\text{crit}} \leq \sigma_p \tag{66}$$

Such range is valid only if the mechanical characteristics of the material are the same as the ones assumed in equations (61), that is the standard quality of concrete. For other

TABLE 1. ($\lambda_{0\,\mathrm{crit}}$). MINIMUM VALUES OF λ_0 AS GIVEN BY EQUATION (61) AND RELATIVE VALUES OF m FOR g $\left(g_{0\,\mathrm{crit}} = \frac{\pi}{24}\frac{E\,.\,10^{-6}}{1-\nu^2}\lambda_{0\,\mathrm{crit}}\right)$

R/L \ $10^2h/R$	0·2	0·4	0·6	0·8	1·0	1·2	1·6	2·0
0·10	0·228 (4)	1·408 (4)	4·494 (4)	10·437 (4)	20·191 (4)	34·708 (4)	81·841 (4)	159·458 (4)
0·20	0·406 (6)	2·733 (6)	7·639 (4)	14·796 (4)	25·904 (4)	41·953 (4)	92·821 (4)	175·304 (4)
0·30	0·625 (8)	3·473 (6)	10·144 (6)	22·739 (6)	43·230 (6)	68·087 (4)	129·920 (4)	225·300 (4)
0·40	0·791 (8)	4·960 (8)	13·051 (6)	26·963 (6)	49·069 (6)	81·421 (6)	185·050 (6)	335·585 (4)
0·50	1·018 (10)	5·709 (8)	17·240 (8)	34·764 (6)	59·557 (6)	95·083 (6)	206·926 (6)	387·467 (4)
0·60	1·174 (10)	6·946 (8)	19·383 (8)	42·575 (8)	76·246 (6)	116·463 (6)	240·020 (6)	436·208 (6)
0·70	1·430 (10)	8·294 (10)	22·573 (8)	47·423 (8)	87·200 (8)	145·592 (8)	286·542 (6)	503·338 (6)
0·80	1·581 (12)	9·189 (10)	26·914 (8)	54·014 (8)	96·653 (8)	158·715 (8)	348·097 (6)	590·999 (6)
0·90	1·779 (12)	10·383 (10)	29·988 (10)	62·599 (8)	108·791 (8)	175·344 (8)	385·826 (8)	700·435 (6)
1·00	2·040 (12)	11·919 (10)	32·798 (10)	71·579 (10)	123·874 (8)	195·808 (8)	421·134 (8)	783·651 (8)
1·25	2·511 (14)	14·692 (12)	42·720 (10)	87·693 (10)	159·009 (10)	263·254 (10)	538·490 (8)	970·463 (8)
1·50	3·048 (16)	18·341 (14)	51·310 (12)	110·814 (10)	193·888 (10)	313·877 (10)	649·127 (10)	1224·465 (8)
1·75	3·641 (16)	21·363 (14)	61·611 (12)	130·337 (12)	239·573 (10)	379·706 (10)	819·258 (10)	1527·082 (10)
2·00	4·169 (18)	25·285 (14)	72·386 (14)	153·063 (12)	227·219 (12)	458·636 (12)	973·934 (10)	1794·446 (10)
2.25	4·810 (18)	28·680 (16)	82·830 (14)	180·585 (12)	321·721 (12)	526·849 (12)	1159·427 (10)	2116·340 (10)
2·50	5·367 (20)	32·582 (16)	95·245 (14)	204·020 (14)	374·092 (12)	607·222 (12)	1346·929 (12)	2496·662 (10)
2·75	6·022 (20)	36·860 (18)	107·575 (16)	231·172 (14)	425·517 (14)	700·275 (12)	1540·808 (12)	2901·718 (12)
3·00	6·648 (22)	42·467 (20)	120·013 (16)	262·183 (14)	478·973 (14)	796·592 (14)	1763·881 (12)	3311·296 (12)

TABLE 2. ($\lambda_{0\,crit}$). MINIMUM VALUES OF λ_0 AS GIVEN BY EQUATION (61) AND RELATIVE VALUES OF m FOR p $\left(p_{0\,crit} = \frac{1}{6}\frac{E\,.\,10^{-6}}{1-\nu^2}\lambda_{0\,crit}\right)$

R/L \ $10^2h/R$	0·2	0·4	0·6	0·8	1·0	1·2	1·6	2·0
0·10	0·186 (4)	1·149 (4)	3·668 (4)	8·519 (4)	16·480 (4)	28·328 (4)	66·798 (4)	97·382 (2)
0·20	0·373 (6)	2·512 (6)	6·235 (4)	12·076 (4)	21·143 (4)	34·242 (4)	75·760 (4)	143·082 (4)
0·30	0·597 (8)	3·191 (6)	9·323 (6)	20·897 (6)	38·467 (4)	55·572 (4)	106·040 (4)	183·889 (4)
0·40	0·755 (8)	4·735 (8)	11·994 (6)	24·779 (6)	45·096 (6)	74·827 (6)	174·622 (4)	273·902 (4)
0·50	0·979 (10)	5·449 (8)	16·458 (8)	31·949 (6)	54·734 (6)	87·384 (6)	190·170 (6)	356·091 (6)
0·60	1·140 (10)	6·631 (8)	18·502 (8)	40·642 (8)	70·072 (6)	107·032 (6)	220·584 (6)	400·855 (6)
0·70	1·389 (10)	8·053 (10)	21·514 (8)	45·269 (8)	83·240 (8)	135·247 (6)	263·339 (6)	462·580 (6)
0·80	1·549 (12)	8·922 (10)	25·692 (8)	51·561 (8)	92·264 (8)	151·508 (8)	319·909 (6)	543·141 (6)
0·90	1·743 (12)	10·082 (10)	29·118 (10)	59·756 (8)	103·851 (8)	167·382 (8)	368·305 (8)	643·716 (6)
1·00	1·999 (12)	11·574 (10)	31·846 (10)	69·501 (10)	118·248 (8)	186·916 (8)	402·010 (8)	748·064 (8)
1·25	2·474 (14)	14·396 (12)	41·480 (10)	85·148 (10)	154·394 (10)	253·065 (8)	514·037 (8)	926·394 (8)
1.50	3·014 (16)	18·070 (14)	50·276 (12)	107·598 (10)	188·261 (10)	304·768 (10)	667·552 (8)	1168·862 (8)
1·75	3·600 (16)	21·047 (14)	60·371 (12)	127·712 (12)	232·620 (10)	368·686 (10)	795·482 (10)	1477·291 (8)
2·00	4·132 (18)	24·911 (14)	71·316 (14)	149·981 (12)	271·637 (12)	447·787 (10)	945·669 (10)	1742·368 (10)
2·25	4·767 (18)	28·356 (16)	81·605 (14)	176·949 (12)	315·242 (12)	516·239 (12)	1125·779 (10)	2054·920 (10)
2·50	5·328 (20)	32·214 (16)	93·836 (14)	201·003 (14)	366·559 (12)	594·993 (12)	1319·803 (12)	2424·205 (10)
2·75	5·979 (20)	36·530 (18)	106·357 (16)	227·753 (14)	419·224 (14)	686·172 (12)	1509·778 (12)	2843·282 (12)
3·00	6·608 (22)	40·437 (18)	118·654 (16)	258·306 (14)	471·889 (14)	784·811 (14)	1728·359 (12)	3244·611 (12)

TABLE 3. (λ_{crit}). MINIMUM VALUES OF λ^* AS GIVEN BY EQUATION (67) AND RELATIVE VALUES OF m FOR g $\left(g_{crit} = \frac{\pi}{24}\frac{E\,.\,10^{-6}}{1-\nu^2}\lambda_{crit}\right)$

R/L \ $10^2h/R$	0·2	0·4	0·6	0·8	1·0	1·2	1·6	2·0
0·10	0·166 (4)	0·949 (4)	2·949 (4)	6·733 (4)	13·029 (4)	22·327 (4)	52·478 (4)	102·091 (4)
0·20	0·292 (6)	1·924 (6)	5·648 (4)	10·565 (4)	18·011 (4)	28·600 (4)	61·698 (4)	114·859 (4)
0·30	0·446 (8)	2·504 (6)	7·188 (6)	15·981 (6)	30·254 (6)	49·330 (4)	91·092 (4)	154·082 (4)
0·40	0·572 (8)	3·528 (8)	9·421 (6)	19·189 (6)	34·622 (6)	57·135 (6)	129·069 (6)	234·855 (4)
0·50	0·718 (10)	4·086 (8)	12·237 (8)	25·063 (6)	42·440 (6)	67·202 (6)	144·783 (6)	269·618 (6)
0·60	0·840 (10)	5·006 (8)	13·805 (8)	30·156 (8)	54·758 (6)	82·840 (6)	168·508 (6)	303·851 (6)
0·70	1·017 (12)	5·885 (10)	16·113 (8)	33·651 (8)	61·612 (8)	102·603 (8)	201·613 (6)	350·781 (6)
0·80	1·122 (12)	6·532 (10)	19·304 (8)	38·399 (8)	68·336 (8)	111·826 (8)	245·001 (6)	411·587 (6)
0·90	1·265 (12)	7·394 (10)	21·211 (10)	44·566 (8)	76·957 (8)	123·508 (8)	270·412 (8)	486·806 (6)
1·00	1·440 (14)	8·502 (10)	23·209 (10)	50·458 (10)	87·642 (8)	137·856 (8)	294·723 (8)	546·642 (8)
1·25	1·774 (14)	10·376 (12)	30·253 (10)	61·760 (10)	111·628 (10)	184·445 (10)	375·209 (8)	673·054 (8)
1·50	2·143 (16)	12·883 (14)	36·055 (12)	77·939 (10)	135·800 (10)	219·243 (10)	483·340 (10)	483·806 (8)
1·75	2·562 (16)	15·001 (14)	43·254 (12)	91·183 (10)	167·365 (10)	264·377 (10)	568·143 (10)	1056·742 (10)
2·00	2·922 (18)	17·750 (14)	50·616 (14)	106·894 (12)	193·128 (10)	319·036 (12)	672·675 (10)	1236·110 (10)
2·25	3·371 (20)	20·060 (16)	57·855 (14)	125·855 (14)	223·635 (12)	365·551 (12)	797·642 (10)	1451·460 (10)
2·50	3·753 (20)	22·777 (16)	66·456 (14)	141·976 (14)	259·486 (12)	420·285 (12)	929·992 (12)	1705·171 (10)
2·75	4·211 (20)	25·714 (18)	74·926 (16)	160·640 (14)	295·184 (14)	483·557 (12)	1060·971 (12)	1995·153 (12)
3·00	4·641 (22)	28·450 (18)	83·512 (16)	181·941 (14)	331·727 (14)	551·047 (14)	1211·462 (12)	2270·503 (12)

TABLE 4. (λ_{crit}). MINIMUM VALUES OF λ^* AS GIVEN BY EQUATION (67) AND RELATIVE VALUES OF m FOR p $\left(p_{crit} = \frac{1}{6}\frac{E\,.\,10^{-6}}{1-\nu^2}\lambda_{crit}\right)$

R/L \ $10^2h/R$	0·2	0·4	0·6	0·8	1·0	1·2	1·6	2·0
0·10	0·134 (4)	0·738 (4)	2·260 (4)	5·156 (4)	9·887 (4)	16·910 (4)	39·670 (4)	58·429 (2)
0·20	0·212 (6)	1·367 (6)	4·402 (6)	8·441 (4)	14·218 (4)	22·360 (4)	47·599 (4)	87·908 (4)
0·30	0·302 (6)	1·819 (6)	5·137 (6)	11·337 (6)	21·380 (6)	36·228 (6)	72·054 (4)	120·457 (4)
0·40	0·391 (8)	2·374 (8)	6·841 (6)	13·758 (6)	24·633 (6)	40·457 (6)	90·908 (6)	173·030 (6)
0·50	0·469 (10)	2·764 (8)	8·216 (8)	18·135 (6)	30·408 (6)	47·818 (6)	102·159 (6)	189·376 (6)
0·60	0·551 (10)	3·405 (8)	9·292 (8)	20·198 (8)	37·796 (8)	59·078 (6)	118·965 (6)	213·235 (6)
0·70	0·649 (12)	3·828 (10)	10·868 (8)	22·557 (8)	41·155 (8)	68·391 (8)	141·993 (6)	245·458 (6)
0·80	0·717 (12)	4·253 (10)	12·721 (10)	25·740 (8)	45·618 (8)	74·454 (8)	166·169 (8)	286·380 (6)
0·90	0·810 (12)	4·818 (10)	13·749 (10)	29·838 (8)	51·298 (8)	82·089 (8)	179·124 (8)	335·838 (6)
1·00	0·907 (14)	5·451 (12)	15·037 (10)	32·604 (10)	58·271 (8)	91·390 (8)	194·697 (8)	360·433 (8)
1·25	1·119 (14)	6·600 (12)	19·478 (12)	39·774 (10)	71·769 (10)	118·463 (10)	245·513 (8)	439·740 (8)
1·50	1·338 (16)	8·087 (14)	22·823 (12)	49·744 (12)	86·848 (10)	140·092 (10)	308·545 (10)	544·928 (8)
1·75	1·595 (18)	9·402 (14)	27·289 (12)	57·470 (12)	105·722 (12)	167·855 (10)	360·585 (10)	670·552 (10)
2·00	1·812 (18)	11·072 (16)	31·617 (14)	67·114 (12)	121·223 (12)	200·221 (12)	424·241 (10)	779·822 (10)
2·25	2·082 (20)	12·458 (16)	36·051 (14)	78·422 (14)	139·820 (12)	228·574 (12)	499·830 (10)	910·360 (10)
2·50	2·316 (20)	14·121 (16)	41·296 (14)	88·233 (14)	161·561 (12)	261·801 (12)	579·608 (12)	1063·574 (10)
2·75	2·597 (20)	15·880 (18)	46·362 (16)	99·575 (14)	183·027 (14)	300·077 (12)	659·037 (12)	1239·932 (12)
3·00	2·854 (22)	17·548 (18)	51·582 (16)	112·481 (14)	205·199 (14)	140·981 (14)	750·136 (12)	1406·921 (12)

materials or other mechanical characteristics, such range has no meaning and it is, therefore, necessary to verify case by case if $\sigma_{crit} \lesseqgtr \sigma_p$.

In case of:

$$\sigma_{crit}^{(0)} > \sigma_p \qquad (66)'$$

where $\sigma_{crit}^{(0)}$ is the maximum comparison tension corresponding to the instability of the shell made of material having undetermined linear elasticity, we can obtain limits inferior to the actual critical load with the procedure indicated in [1]. Such procedure essentially consists of substituting in (64) the initial elasticity modulus E with the tangential modulus for the shell:

$$E_t = E\frac{\sigma_{crit}}{\sigma_{crit}^{(0)}} \qquad (67)$$

and the approximate value of σ_{crit} can be obtained from:

$$\sigma_{crit} = K_1 - \pi K_2\sqrt{\frac{E}{\sigma_{crit}^{(0)}}} \qquad (68)$$

where K_1 and K_2 are the material constants of the well-known Tetmayer equation:

$$\sigma_{crit} = K_1 - K_2\frac{l}{\rho_{min}} \qquad (69)$$

for the instability of short columns.

For concrete the authors [1] suggest:

$$\left.\begin{aligned} K_1 &= 4000\ \text{ton/m}^2 \\ K_2 &= 21{\cdot}79\ \text{ton/m}^2 \end{aligned}\right. \qquad (70)$$

(K_1 is also the yield stress of the material).

In every case we must verify for reinforced concrete shells if, prior to instability, we have a collapse due to failure of the reinforcement provided to absorb the tensile stresses.

6. TWO SPECIAL CASES

From the studies performed by Jakobsen (see [1]) it appears that most shells realized so far can be classified under two types in accordance with the following geometrical ratios:

$$\text{Type}\begin{cases} \text{I, long: } R/L = 0{\cdot}36 \quad h/R = 0{\cdot}694 \times 10^{-2} \\ \text{II, short: } R/L = 2{\cdot}70 \quad h/R = 0{\cdot}300 \times 10^{-2} \end{cases} \qquad (71)$$

For such types we give in Tables 5 and 6 the numerical results of performed calculations, directly compared with the results obtained in [1], in order to clearly show the differences obtained with the proposed procedure. (Results always apply to the case $\nu = 0$).

TABLE 5. MINIMUM VALUES λ_{crit} FOR TYPE I SHELLS

Type I	In accordance with [1]		In accordance with proposed theory for $\eta = 0$			In accordance with proposed theory for η (min.)		
	λ_{crit}	m	λ_{crit}	m	η	λ_{crit}	m	η
Load g	16·929	6	16·998	6	0	12·098	6	0·702
Load p	16·004	6	15·621	6	0	8·679	6	1·271

Figures 3 and 4 show the directrix buckled shapes obtained in accordance with numerical data of Tables 5 and 6 in conformity with expression (28).

In drawing the buckled shapes we used only the normal component w, neglecting the effect of tangential component v. These figures clearly confirm the importance of the waves damping in the buckled shape of neutral equilibrium C_1 for long shells as well as for short ones.

TABLE 6. MINIMUM VALUES λ_{crit} FOR TYPE II SHELLS

Type II	In accordance with [1]		In accordance with proposed theory for $\eta = 0$			In accordance with proposed theory for η (min.)		
	λ_{crit}	m	λ_{crit}	m	η	λ_{crit}	m	η
Load g	18·041	20	16·848	18	0	11·775	18	1·213
Load p	17·955	20	16·697	18	0	7·275	18	2·544

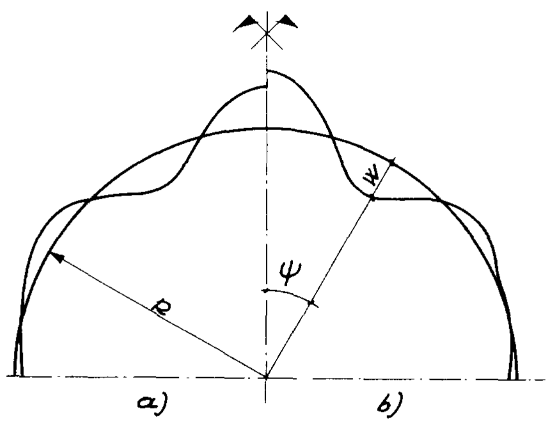

FIG. 3. Buckling shape of long cylindrical shell roof (a) in case of dead load (b) in case of live load.

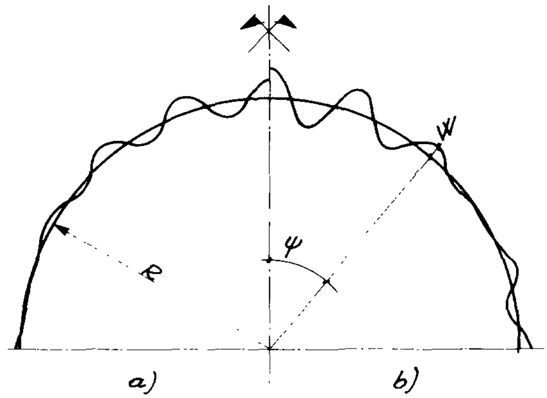

FIG. 4. Buckling shape of short cylindrical shell roof (a) in case of dead load (b) in case of live load.

7. NUMERICAL EXAMPLES

The following numerical examples are an application of what was mentioned above:

(a) *Reinforced concrete shells having:*

$$R = 10\text{ m} \qquad L = 50\text{ m} \qquad h = 4\text{ cm.}$$

We have:

$$R/L = 0{\cdot}2 \qquad h/R = 0{\cdot}4 \times 10^{-2}$$

therefore, from Tables 3 and 4:

$$\lambda_{g\,\text{crit}} = 1{\cdot}924$$

$$\lambda_{p\,\text{crit}} = 1{\cdot}367.$$

Assuming the concrete mechanical characteristics to be those expressed by (65), we have:

$$g_{\text{crit}} = \frac{\pi}{24} \times \frac{2 \times 10^6 \times 10^{-6}}{1 - 0{\cdot}04} \times 1{\cdot}924 = 0{\cdot}525 \text{ ton/m}^2$$

$$p_{\text{crit}} = \frac{1}{6} \times \frac{2 \times 10^6 \times 10^{-6}}{1 - 0{\cdot}04} \times 1{\cdot}367 = 0{\cdot}475 \text{ ton/m}^2.$$

For stresses σ_x corresponding to the generatrix of each cross center section we have from (45) and (50):

$$\sigma_{xg\,\text{crit}} = -\frac{g_{\text{crit}} L^2}{4Rh} = -820 \text{ ton/m}^2$$

$$\sigma_{xp\,\text{crit}} = -\frac{3p_{\text{crit}} L^2}{8Rh} = -1113 \text{ ton/m}^2$$

therefore, as can be deduced from the tables, in both cases instability takes place in a proportional elastic range.

If we want to consider the simultaneous presence of g and p, having fixed g and variable p, and determine the load p_e corresponding to the shell unstability, we can assume as a first approximation:

$$p_e = p_{crit}\left(1-\frac{g}{g_{crit}}\right) \tag{72}$$

which corresponds to the linear expression of boundary curve of stability field p, g.

In our case, assuming

$$g = 0{\cdot}130 \text{ ton/m}^2$$

we have, from (72)

$$p_e = 0{\cdot}475\left(1-\frac{0{\cdot}130}{0{\cdot}525}\right) = 0{\cdot}357 \text{ ton/m}^2.$$

(b) *Reinforced concrete shell having geometrical characteristics:*

$$R = 9\text{ m} \qquad L = 25\text{ m} \qquad h = 6{\cdot}25\text{ cm.}$$

We have:

$$R/L = 0{\cdot}36 \qquad h/R = 0{\cdot}094\times 10^{-2}$$

typical of long shells.

From Table 5 we have:

$$\lambda_{g\,crit} = 12{\cdot}098$$
$$\lambda_{p\,crit} = 8{\cdot}679$$

and assuming concrete mechanical characteristics as per (65):

$$g_{crit} = 3{\cdot}30 \text{ ton/m}^2$$
$$p_{crit} = 3{\cdot}016 \text{ ton/m}^2$$

we have

$$\sigma_{xg\,crit} = 917 \text{ ton/m}^2$$
$$\sigma_{xp\,crit} = -1257 \text{ ton/m}^2.$$

If we apply equation (72), assuming that

$$g = 0{\cdot}190 \text{ ton/m}^2$$

we have

$$p_e = 3{\cdot}016\left(1-\frac{0{\cdot}190}{3{\cdot}301}\right) = 2{\cdot}934 \text{ ton/m}^2.$$

8. CONCLUSIONS

We can conclude by saying that the instability of self-supporting cylindrical shell roofs occurs by buckling of the directrix with consequent formation of various lobes which damp out when proceeding from the compressed to the tensile areas.

Such a phenomenon can occur in the elastic as well as in the inelastic range, depending upon the values of the following ratios: thickness/radius, h/R, and radius/span, R/L, as well as the proportionality limit σ_p. However, in many cases the critical stresses σ_{crit} are very high, and therefore they have no meaning if the changes of elastic modulus E are not taken into proper account.

In the inelastic range we do obtain acceptable values of the critical loads through the considerations mentioned at the end of Section 5 above; however, they must be considered as approximate values and treated accordingly.

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Résumé—Dans cette thèse nous étudions la détermination des charges critiques au cas de toits à carcasse cylindrique isolés et indépendants, soumis à des charges mortes at à des charges vives uniformes. Il semble être nécessaire d'introduire dans la direction circonférentiel une forme de flambement plus compliqué que de simples ondes sinusoidales, et le problème se résout par la méthode d'énergie tout en se servant d'une hypothèse simplifiée de la théorie de Donnell.

Afin de pouvoir appliquer cette théorie nous présentons des tables qui permettront de calculer les valeurs des charges critiques directement dans plusieurs cas. Les résultats sont plus petits que les valeurs correspondantes obtenues dans une étude récente. Quelques exemples numériques concluent cette étude.

Zusammenfassung—Die Arbeit behandelt die Bestimmung kritischer Belastungen für freie selbsttragende Schalendächer unter ruhender und gleichmäßiger Last. Es scheint notwendig in der Umgehrichtung eine Knickform einzuführen die komplizierter ist als eine einfache Sinuslinie, die Lösung erfolgt mittels einer vereinfachten Hypothese der Theorie von Donnell. Um diese Theorie anzuwenden wird eine Zahlentafel gegeben, die eine einfache Errechnung in vielen Fällen ermöglicht. Die Resultatswerte sind geringer als die Werte die in einer ähnlichen Arbeit vor einiger Zeit erhalten wurden. Einige errechnete Beispiele beenden diese Arbeit.

Абстракт—В этой статье мы изучаем деформацию критических нагрузок для изолированных самоподпирающихся цилиндрических крыш в виде свода оболочки под мёртвыми нагрузками и однородными временными нагрузками. Оказалось необходимым ввести в смысле догадки форму изгиба более сложную, чем простые синусоидальные волны и проблема разрешается методом энергии с применением упрощённой гипотезы теории Доннелла (Donnell's). Чтобы применить зту теорию мы предлагаем таблицы, которые во многих случаях позволяют прямо расчитать значение критических нагрузок. Результаты меньше, чем соответствующие значения критических нагрузок, полученные недавним исследованием (I). Несколько цифровых примеров заканчивают эту статью.